Do vector plane waves form complete basis of solutions to Maxwell's equations? Introduction to Generalized Plane Wave Solutions

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Abstract

As the title says, this article questions the well established belief that the vector plane waves form a complete basis of solutions to Maxwell's equations. Vector solutions to Maxwell's equations are presented here, which have planar wave-fronts and transverse electric and magnetic fields but spatially varying polarization. They form a one-parameter family specified by integer n, and are termed as *generalized vector plane waves*. The known vector plane wave solution with spatially uniform polarization, referred to as *conventional vector plane waves* in this article, is a subset of this family obtained for $n = 0$. In contradiction to the established belief, it is shown that these *generalized vector plane waves* with spatially varying polarization (for $n \neq 0$) cannot be expressed as superposition of conventional vector plane waves. The family of solutions also includes the interesting cases of radially and azimuthally polarized plane waves for $n = 1$.

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I. INTRODUCTION

Vector Plane waves with polarization of uniform magnitude and direction are the simplest known solution to Maxwell's equations. Owing to mathematical complexity of full vector Maxwell's equations however, most problems in classical optics are treated with scalar wave equation. In Fourier optics, scalar plane waves are natural mode of propagation in free space, and any scalar electromagnetic field solution can be represented as superposition of plane waves, if the Fourier Transform of the solution exists. Analogously, it has been generally assumed in the scientific community that the vector plane waves form a complete basis for all solutions to vector Maxwell's equations. In this letter, a new class of vector solutions to Maxwell's equations is presented, which cannot be expressed as superposition of vector plane waves. The derivation is done without making any approximations and in vacuum region free of charge and current. These solutions form a one-parameter family specified by integer n, and termed as *generalized vector plane waves* since they possess planar wave-front but has spatially varying polarization, including radial and azimuthal polarization distributions. To avoid confusion, the known vector plane wave solution with spatially uniform polarization shall be referred as *conventional* vector plane waves whereas the new solution being presented in this paper shall be referred as *generalized* vector plane waves. Interestingly, the conventional vector plane wave solution itself is a special case of the family of *generalized plane wave solutions*, which is obtained for $n = 0$.

II. THEORY

The form of Maxwell equations in free space can be written as follows:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}\right) \tag{4}$$

In a region free of charge and current, $\rho = \mathbf{J} = 0$.

The Maxwell's equations then reduce to wave equations in $\mathbf{E}$ under the Lorentz gauge condition [1] ,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0 \tag{5}$$

where $c = \frac{1}{\sqrt{\mu_0 \in_0}}$ is the speed of light in vacuum.

Assuming time-harmonic form of the solution, we can write,

$$\mathbf{E}(r,\theta,z,t) = \tilde{\mathbf{E}}(r,\theta,z)e^{-i\omega t} \tag{6}$$

,

where ω is the frequency of the wave, and (r,θ,z) represent the cylindrical coordinate system. Substituting the above time harmonic electric field into (5), we obtain the Helmholtz equation in $\tilde{\mathbf{E}}$

$$\nabla^2 \tilde{\mathbf{E}} + k^2 \tilde{\mathbf{E}} = 0 \tag{7}$$

where $k = \frac{\omega}{c}$ is the wavenumber. As we would only focus on the time-independent part of the field, the tilde from $\tilde{\mathbf{E}}$ will be dropped in subsequent analysis. $\mathbf{E}$ shall be understood as time-independent part of the electric field.

To seek non-diffracting generalized plane wave solutions (cite paper), we assume an ansatz of the time-independent electric field $\mathbf{E}(r,\theta,z)$ with zero longitudinal component and the transverse components dependent only on (r,θ).

$$\mathbf{E}(r,\theta,z) = (E_1(r,\theta)\mathbf{e}_1 + E_2(r,\theta)\mathbf{e}_2)\exp[ikz] \tag{8}$$

We have expressed the electric field ansatz in the Cartesian vector basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ but we will solve the Maxwell's equations in cylindrical coordinates (r,θ,z).

In cylindrical coordinates, the Laplacian operator, ∇^2 is given by,

$$\begin{aligned}\nabla^2\mathbf{E} &= \frac{\partial^2\mathbf{E}}{\partial r^2} + \frac{1}{r}\frac{\partial\mathbf{E}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\mathbf{E}}{\partial\theta^2} + \frac{\partial^2\mathbf{E}}{\partial z^2}\\ &= \frac{\partial^2\mathbf{E}}{\partial r^2} + \frac{1}{r}\frac{\partial\mathbf{E}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\mathbf{E}}{\partial\theta^2} - k^2\mathbf{E}\\ \Rightarrow \nabla^2\mathbf{E} + k^2\mathbf{E} &= \frac{\partial^2\mathbf{E}}{\partial r^2} + \frac{1}{r}\frac{\partial\mathbf{E}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\mathbf{E}}{\partial\theta^2}\end{aligned} \tag{9}$$

The right hand side of the above equation can be looked as transverse part of the Laplacian operator, ∇_t^2, while the left hand side is zero using equation (7). Thus, we get

$$\begin{aligned} \nabla_t^2 \mathbf{E} &= 0 \\ \text{or,} \quad \nabla_t^2 E_i &= 0 \end{aligned} \tag{10}$$

where $i = 1, 2$ correspond to the x and y components of electric field respectively. Each vector component of electric field E_i satisfies Laplacian equation in polar coordinates(r, θ), whose solution is known to be of the following form [2]

$$E_i(r, \theta) = r^n (a_{in} \cos(n\theta) + b_{in} \sin(n\theta)) \tag{11}$$

where n can be any integer and a_{in}, b_{in} are constants to be determined from divergence free condition and boundary conditions.

Divergence free condition

$$\nabla \cdot \mathbf{E} = \mathbf{0} \tag{12}$$

$$\Rightarrow \left(E_{1,r} + \frac{E_{2,\theta}}{r} \right) \cos\theta + \left(E_{2,r} - \frac{E_{1,\theta}}{r} \right) \sin\theta = 0 \tag{13}$$

Substituting $E_1(r, \theta)$ and $E_2(r, \theta)$ from equation (11) and simplifying using trigonometric identities, we get,

$$nr^{n-1}((a_{1n} + b_{2n}) \cos(n-1)\theta + (b_{1n} - a_{2n}) \sin(n-1)\theta) = 0 \tag{14}$$

Since equation (14) must hold true for all values of r and θ, the coefficients of each term are equated to zero.

$$\begin{aligned} a_{1n} &= -b_{2n} \\ b_{1n} &= a_{2n} \end{aligned} \tag{15}$$

Substituting these coefficients from the above relations back into equation (11), we get,

$$\begin{aligned} E_{1n} &= r^n (a_{1n} \cos(n\theta) + a_{2n} \sin(n\theta)) \\ E_{2n} &= r^n (-a_{1n} \sin(n\theta) + a_{2n} \cos(n\theta)) \end{aligned} \tag{16}$$

Let $\mathbf{R}(\psi)$ be a rotation matrix which rotates any vector by angle ψ counter-clockwise about $z-$ axis. Then the above solution can be written more elegantly as,

$$\mathbf{E}^n(r,\theta,z) = r^n \mathbf{R}(-n\theta) \cdot \mathbf{A} \exp(ikz) \tag{17}$$

where $\mathbf{A} = (a_{1n}, a_{2n}, 0)$ is a constant vector. This completes the derivation of *generalized plane waves*. On substituting $n = 0$, we recover the "conventional" plane wave solution.

$$\mathbf{E}^0(r,\theta,z) = \mathbf{A} \exp(ikz) \tag{18}$$

The vector $\mathbf{A}$ thus, defines the polarization of the conventional vector plane wave $\mathbf{E}^0$. Analogously, vector $\mathbf{A}$ is defined to be the polarization of the generalized plane wave solution for all n as well. Note that throughout our analysis, the constants a_{1n}, a_{2n} can be complex numbers, as such the notions of linearly, circularly or elliptically polarization extend to the generalized plane waves as well.

Unlike conventional vector plane waves, the generalized plane waves can exist only in certain domains depending on the sign of the charge n owing to the amplitude's dependence on r^n. When $n > 0$, the amplitude blows up as $r \to \infty$ and hence $\mathbf{E}^n(\mathbf{x})$ for any $n > 0$ cannot exist in an unbounded domain. Figure (1) shows the electric field given by the solution (17) with $\mathbf{A} = (1,0,0)$ and $n = 1$. The electric field vector rotates in the opposite sense with respect to θ, which is the case for all $n > 0$ and hence we term these solutions as $counter-rotating$ generalized plane waves.

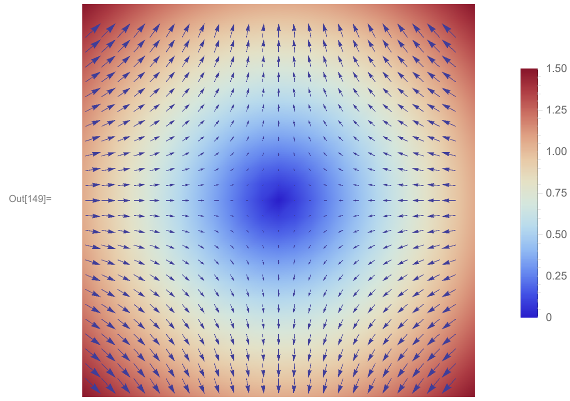

FIG. 1: *Counter-rotating* generalized plane wave solution given by equation (17) with $\mathbf{A} = (1, 0, 0)$ and $n = 1$. The amplitude of the field vectors is represented by the arrow length as well as the background color (color scale in plot legend)

When $n < 0$, the amplitude blows up as $r \to 0$ and therefore, in this case, $\mathbf{E}^n(\mathbf{x})$ can only exist in a domain excluding the z-axis $r = 0$, such as region outside a cylinder. Figure (2) shows the electric field solution given by equation (17) with $n = -1$ and polarization given by $\mathbf{A} = (1, 0, 0)$, which has radial polarization distribution. Since the electric field vector rotates in the same sense as the azimuthal angle θ when $n < 0$, these solutions are termed as $co - rotating$ generalized plane waves. For the same $n = -1$ but $\mathbf{A} = (0, 1, 0)$, azimuthally polarized electric field is produced as seen in figure (3). In both figures (2) and (3), a small region near the axis $r = 0$ (not visible in figures) has been excluded to keep the amplitude finite.

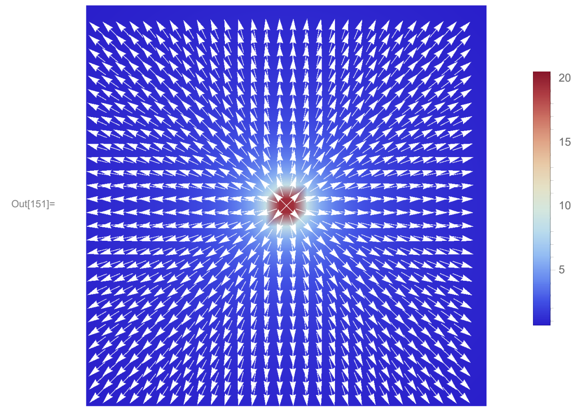

FIG. 2: *Co-rotating* generalized polarized plane wave with $n = -1$ and $\mathbf{A} = (1, 0, 0)$ has radial polarization distribution. The background color represents the magnitude of field vectors, while the arrows are of constant length indicating the direction of field.

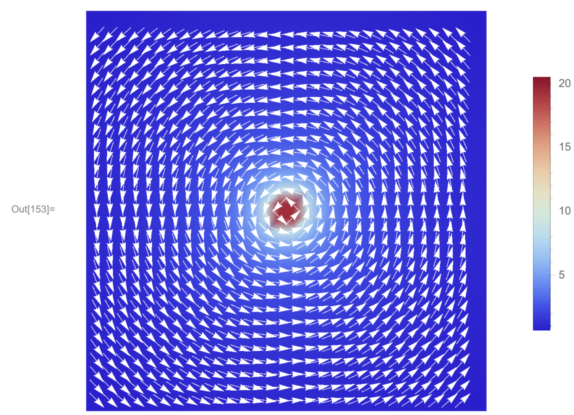

FIG. 3: *Co-rotating* generalized polarized plane wave with $n = -1$ and $\mathbf{A} = (0, 1, 0)$ has azimuthal polarization distribution. The background color represents the magnitude of field vectors, while the arrows are of constant length indicating the direction of field.

III. THEOREM ON ORTHOGONALITY

The expression, given by equation (17) of *generalized plane waves* traveling in z-direction has polarization spatially varying in transverse plane for $n \neq 0$. Indeed, it cannot be expressed as superposition of any finite or infinite set of *conventional plane waves* traveling in z-direction, since the sum of all of them would still be a plane wave having polarization constant in transverse plane. Thus, our only hope is to check if the *generalized plane waves* can be approximated by *conventional plane waves* traveling in different directions. A formal proof by contradiction is presented here showing that the *generalized plane waves* with $n \neq 0$ cannot be expressed as superposition of conventional plane wave, which is also subset of the former with $n = 0$.

Lets assume that generalized plane waves $\mathbf{E}^n(\mathbf{x})$ given by equation (17) can be approximated by M conventional plane waves as written below:

$$\mathbf{E}^n(\mathbf{x}) = r^n \mathbf{R}(-n\theta) \cdot \mathbf{A} \exp(ikz) = \sum_{m=1}^{M} \mathbf{p}_m \exp(i\mathbf{k}_m \cdot \mathbf{x}) \tag{19}$$

where $\mathbf{k}_m = (k_{mx}, k_{my}, k_{mz})$ is the wave-vector of the mth plane wave and $\mathbf{p}_m$ its polarization such that $\mathbf{p}_m \cdot \mathbf{k}_m = 0$. Note that,

$$\mid \mathbf{k}_m \mid = k \text{ for all } m \tag{20}$$

stemming from the fact that all the plane waves involved in superposition (19) must be of the same frequency (which implies in free space, same wave-number too), each satisfying the same Helmholtz equation (7).

Equation (19) has harmonic functions in z variable on either side, which is why we will take Fourier Transform in z of the whole equation. Multiplying equation (19) by $\exp(-ilz)$ and integrating over z domain, we obtain,

$$\int_{-\infty}^{\infty} r^n \mathbf{R}(-n\theta) \cdot \mathbf{A} \exp(ikz) \quad \exp(-ilz) \quad dz = \int_{-\infty}^{\infty} \sum_{m=1}^{M} \mathbf{p}_m \exp(i\mathbf{k}_m \cdot \mathbf{x}) \quad \exp(-ilz) \quad dz$$

$$\Rightarrow r^n \mathbf{R}(-n\theta) \cdot \mathbf{A} \delta(k-l) = \sum_{m=1}^{M} \int_{-\infty}^{\infty} \mathbf{p}_m \exp(i(k_{mx}x + k_{my}y + k_{mz}z) \quad \exp(-ilz) \quad dz$$

$$\Rightarrow r^n \mathbf{R}(-n\theta) \cdot \mathbf{A} \delta(k-l) = \sum_{m=1}^{M} \mathbf{p}_m \exp(i(k_{mx}x + k_{my}y)) \quad \delta(k_{mz} - l) \tag{21}$$

Now, let's define a set $\mathcal{S}$ of plane waves given by index m such that $k_{mz} = k$.

$$\mathcal{S} = \{1 \leq m \leq M \mid k_{mz} = k\} \tag{22}$$

Further, for any $m \in \mathcal{S}$, equation (20) implies

$$\begin{aligned} k_{mx}^2 + k_{my}^2 &= 0 \\ \Rightarrow\ k_{mx} = k_{my} &= 0 \end{aligned} \tag{23}$$

since the wave-vectors of plane waves used in the superposition (19), being in free space, cannot be imaginary or complex numbers.

In parallel, we define the complimentary set $\mathcal{S}'$ as

$$\mathcal{S}' = \{1 \leq m \leq M \mid k_{mz} \neq k\} \tag{24}$$

For any $m \in \mathcal{S}'$, substituting $l = k_{mz}$,in equation (21) we get,

$$0 = \mathbf{p}_m \exp(i(k_{mx}x + k_{my}y)) \quad \Rightarrow \mathbf{p}_m = \mathbf{0} \tag{25}$$

Equation (21) thus tells us that the plane waves corresponding to index $m \in \mathcal{S}'$ must have zero contribution to the superposition (19). The above equation (25) was written assuming that there is only one plane wave with that particular value of k_{mz}. If there are more than one plane waves with the same value of k_{mz}, still their net contribution to the superposition (19) would be zero.

On the other hand, for any $m \in \mathcal{S}$, substituting $l = k = k_{mz}$ in equation (21), we get,

$$\begin{aligned} r^n \mathbf{R}(-n\theta) \cdot \mathbf{A} &= \sum_{m \in \mathcal{S}} \mathbf{p}_m \exp(i(k_{mx}x + k_{my}y)) \\ &= \sum_{m \in \mathcal{S}} \mathbf{p}_m \exp(i(k_{mx}r\cos\theta + k_{my}r\sin\theta)) \end{aligned} \tag{26}$$

where the Cartesian coordinates (x, y) were written in the form of cylindrical coordinates as $(r\cos\theta, r\sin\theta)$ in the last line. Finally, using (23), we get,

$$r^n \mathbf{R}(-n\theta) \cdot \mathbf{A} = \sum_{m \in \mathcal{S}} \mathbf{p}_m \tag{27}$$

The left hand side of the above equation (27) is a function of (r, θ), while the right hand side is a sum of constant vectors, which itself, must be a constant vector, which clearly is a contradiction. This completes the proof!

More generally, it's an easy proof that any two generalized vector plane waves with different charge n traveling along the same direction, here $z-$ axis are orthogonal. However, it is not trivial to check the orthogonality of any two generalized plane waves with different charge traveling in different direction. The expression for generalized plane wave with $\mathbf{k}$ not along z-axis can be written in a different cylindrical coordinate system (r', θ', z') where z' is chosen along $\mathbf{k}$ and r' and θ' are chosen appropriately so that the field solution can be given by the same expression as in (17) but with (r, θ, z) replaced by (r', θ', z'). While computing the Fourier transform with respect to z variable of a field traveling along $z'-$ direction, (r', θ', z') would need to be expressed as function of (r, θ, z). The Fourier transform may no longer be a simple delta function as in equation (21), which makes it difficult to check the orthogonality.

Another important remark is that the restriction on n to be integer, comes from the circular symmetry of the domain (here $\mathbb{R}^3$). The generalized plane waves can still exist in non-circular symmetric domains. The restriction on n to be an integer would be relaxed in that case but additional boundary conditions would be required instead to determine possible values of n.

IV. CONCLUSIONS

In summary, it has been proved that the *generalized vector plane waves* cannot be expressed as superposition of *conventional vector plane waves*. The former constitutes a larger set of solutions to full vector Maxwell's equations, of which the later is a subset. The belief that conventional plane wave solutions form a complete basis of solutions to vector Maxwell's equations is not true. Any other vector solutions basis, such as one presented in [3], shown to be complete based on this belief is not a complete basis either. It remains to be seen whether the basis formed by the generalized plane waves is complete, and also whether they are mutually orthogonal for different $\mathbf{k}$ and n.

The existence of vector solutions to Maxwell's equations, which cannot be expressed as superposition of *conventional vector plane waves* has profound implications on the conventional mathematical techniques of Fourier Optics. If any optical element can stimulate generation of these *generalized plane waves* in free space, their diffraction cannot be studied by the theory of Fourier Optics, based on the angular spectrum of scalar plane waves. Vector theories of diffraction incorporating at least the *generalized vector plane waves* solution need to formulated for correct systematic mathematical study of such problems.

How these generalized plane waves can be experimentally realized is another interesting open question. In subsequent papers, some of my theoretical attempts at designing experiment to realize simpler types of generalized plane waves will be presented. Recently, optical beams of spatially varying polarization of different kinds have gained attention and been experimentally realized. The vector solutions presented here along with other known vector solutions to Maxwell's equation [3–5] should be instrumental in the development of better theoretical understanding of experimental research in vector optics.

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